# Infinite Products and Mittag-Leffler Expansion

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# 1 Abstract

In complex analysis, various infinite series representations such as Taylor Series and Laurent Series are studied. In application, it is often beneficial to represent functions that are analytic in certain domains using infinite products. In this paper, we introduce the basics of infinite products and then focus on three primary theorems: Weierstrass M-test, Mittag-Leffler Expansion and Weierstrass Factorization theorem. These three theorems demonstrate some properties of infinite products. This paper, then, elaborates on the history, significance and applications of infinite products and, in particular, Mittag-Leffler Expansion.

# 2 Intro

In this paper we investigate three theorems in order to explain infinite product applications in complex analysis as well as explain the history and significance. Similar to Taylor or Laurent Series (infinite sums), it is useful to represent a function with an infinite product. The central ideas of infinite products are focused around three theorems: Weierstrass M-test, Mittag-Leffler Expansion and Weierstrass Factorization Theorem.

The Weierstrass M-test for infinite products works towards the same conclusion as the Weierstrass M-test for infinite series. The Weierstrass M-test allows us to determine if an infinite product converges uniformly. This information can be found in Section 3.

The Mittag-Leffler Expansion is concerned with the existence of meromorphic functions and uniform convergence of the functions. More information on Mittag-Leffler can be found in Section 4.

The Weierstrass Factorization theorem discusses entire functions with isolated zeros. This is also called the Weierstrass Factor theorem and is a direct result of the Mittag-Leffler theorem. This theorem is discussed in more detail in Section 5.

Before discussing the theorems, it is important to have a basic understanding of infinite products.

## 2.1 What are Infinite Products?

The simplest definition of an infinite product is that it is a function that can be represented as a product with an infinite number of terms. Like the commonly used series, Taylor and Laurent, infinite products are another way to represent functions of infinite series. Such a product will either converge or diverge.

Let's begin with a sequence of complex numbers, $a_k$. Then an infinite product is denoted by

$$P = \prod_{k=1}^{\infty}(1 + a_k)$$

The above infinite product converges if the sequence of partial products $P_n$,

$$P_n = \prod_{k=1}^{n}(1 + a_k)$$

converges to a finite limit, and for $N_0$ large enough

$$\lim_{N\to\infty} \prod_{k=N_0}^{N} (1 + a_k) \neq 0$$

It is important to note that if $\lim_{N\to\infty} \prod_{k=N_0}^{N}(1 + a_k) = 0$ for all $N_0$, then the product diverges. This is because the infinite sum $S = \sum_{k=1}^{\infty} \log(1 + a_k)$ is connected to the product and it simply would not make sense if $P = 0$. [5]

A simple example of a convergent infinite product is the infinite product $\prod_{i=1}^{\infty} a_n$ converges to a nonzero number if and only if $\prod_{i=1}^{\infty} \ln(a_n)$ converges. [2]

## 2.2 Significance and History

It would be amiss to discuss the history of the Mittag-Leffler Expansion and not begin with Karl Weierstrass. Weierstrass became fascinated with elliptic functions at a local academy in Munster in 1839. When he went on to teach, he began researching the topic in his free time. After teaching various courses, in 1863 he became aware that complex analysis needs to be more rigorously studied. That is when he began developing definitions and theories.

To Weierstrass, uniform convergence was significantly important. When he lectured, he heavily emphasized the difference between convergence and uniform convergence. Weierstrass noted that the key feature of uniform convergence was that if a series of analytic functions are uniformly convergent, then the sum of such functions is analytic.

Weierstrass published his paper on the theory of analytic functions in 1876, one year after Mittag-Leffler studied with him. In this paper, he discussed that since the Fundamental Theorem of Algebra helps prove that a polynomial can be broken into linear factors (one for each zero), then it must be possible to write a "representation of other entire functions as a product of factors in a way that reveals their zeros." [7] From this, he wondered if given an infinite sequence of constants with the limit of terms approaching infinity, if there is a way to always find a function whose zeros are given by the sequence.

While Weierstrass was investigating these questions, Mittag-Leffler began elaborating on Weierstrass's representation theorems of entire functions. He began exploring the applications of these concepts with meromorphic functions. Mittag-Leffler looked at Weierstrass's Factorization theorem, but, instead of looking at polynomials, he looked at rational functions and attempted to generalize to meromorphic functions. He studied whether a function is uniquely specified by its singular points and the coefficients of its Laurent series. He stressed that the "product expansion gives the general analytic expression of a function of entire character whose zeros and their respective orders are prescribed arbitrarily (provided that in any finite neighborhood there is at most a finite number of zeros)." [7] Mittag-Leffler then looked into the representation of functions with an infinite set of essential singular points.

Given this historical context, the theorems and their proofs are presented in the following sections.

# 3 Weierstrass M-test

## 3.1 What is the Weierstrass M-test?

The Weierstrass M-test is used to determine if a product of functions is uniformly convergent. The theorem utilizes two different hypothesis to arrive at the conclusion that the product is uniformly convergent to an analytic function. This is an obviously useful tool; confirming the infinite product is uniformly convergent opens the door to many other applications in complex variables.

## 3.2 Statement

Let the series $a_k(z)$ be analytic in a domain $D$ for all $k$. Suppose for all $z \in D$ and $k \geq N$ either

$$|\log(1 + a_k(z))| \leq M_k \tag{1}$$

or

$$|a_k(z)| \leq M_k \tag{2}$$

where $\sum\limits_{k=1}^{\infty} M_k < \infty$, $M_k$ are constants. Then the product

$$P(z) = \prod_{k=N}^{n} (1 + a_k(z))$$

is uniformly convergent to an analytic function $P(z)$ in $D$. Furthermore, $P(z)$ is only zero when a finite number of its factors $1 + a_k(z)$ are zero in $D$. [5]

### 3.3 Proof

For $n \leq N$, define

$$P_n(z) = \prod_{k=N}^{n} (1 + a_k(z))$$

$$S_n(z) = \sum_{k=N}^{n} \log(1 + a_k(z))$$

Using the first inequality, equation (1), from the hypothesis of the Weierstrass M-test for any $z \in D$ with $m > N$ yields

$$|S_m(z)| \leq \sum_{k=N}^{m} M_k \leq \sum_{k=1}^{\infty} M_k = M < \infty$$

Similarly, for any $z \in D$ with $n > m \geq N$ we have

$$|S_n(z) - S_m(z)| \leq \sum_{k=m+1}^{n} M_k \leq \sum_{k=m+1}^{\infty} M_k \leq \epsilon_m$$

where $\epsilon_m \to 0$ as $m \to \infty$, and $S_n(z)$ is a uniformly convergent Cauchy sequence. Additionally, we know $P_k(z) = \exp S_k(z)$, which tells us

$$(P_n(z) - P_m(z)) = e^{S_m(z)}(e^{S_n(z)-S_m(z)} - 1)$$

From the Taylor series, $e^w = \sum\limits_{n=0}^{\infty} \frac{w^n}{n!}$, we have

$$|e^w| \leq e^{|w|},$$

$$|e^w - 1| \leq e^{|w|} - 1$$

where upon from the above estimates we have

$$|P_n(z) - P_m(z)| \leq e^M(e^{\epsilon_m} - 1)$$

Therefore, $P_n(z)$ is a uniform Cauchy sequence. Then $P_n(z) \to P(z)$ uniformly in $D$ and $P(z)$ is analytic because $P_n(z)$ is a sequence of analytic functions. Moreover, we have

$$\begin{aligned} |P_m(z)| &= |e^{\text{Re}S_m(z)+i\text{Im}S_m(z)}| \\ &= |e^{\text{Re}S_m(z)}| \\ &\geq e^{-|\text{Re}S_m(z)} \\ &= e^{-|\text{Re}S_m(z)+i\text{Im}S_m(z)|} \\ &\geq e^{-M} \end{aligned}$$

thus $P_m(z) \geq e^{-M}$. Note that $P(z) \neq 0$ in $D$ because $M$ is independent of $m$.

Because we may write $P(z) = \prod_{k=1}^{N-1}(1 + a_k(z))\tilde{P}(z)$, we see that $P(z) = 0$ only if any of the factors $(1 + a_k(z)) = 0$, for $k = 1, 2, ..., N-1$. (The estimate (1) of the Weierstrass M-test is invalid for such a possibility.) The analyticity of $a_k(z)$ leads us to

$$P_n(z) = \prod_{k=1}^{N-1}(1 + a_k(z))\tilde{P}_n(z)$$

which is a uniformly convergent sequence of analytic functions. Finally, we note that the first hypothesis, (1), follows from the second hypothesis, (2), as is shown next. The Taylor series of $\log(1 + w)$, $|w| < 1$, is given by

$$\log(1 + w) = (w - \frac{w^2}{2} + \frac{w^3}{3} + ... + (-1)^{n-1}\frac{w^n}{n} + ...)$$

so

$$|\log(1 + w)| \leq |w| + \frac{|w|^2}{2} + \frac{|w|^3}{3} + ... + \frac{|w|^n}{n} + ...$$

and for $|w| \leq \frac{1}{2}$ we have

$$\begin{aligned} |\log(1 + w)| &\leq |w|(1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^n} + ...) \\ &\leq |w|(\frac{1}{1 - \frac{1}{2}} \\ &= 2|w| \end{aligned}$$

therefore, for $|a_k(z)| \leq \frac{1}{2}$,

$$|\log(1 + a_k(z))| \leq 2|a_k(z)|$$

If we assume that $|a_k(z)| \leq M_k$, with $\sum_{k=1}^{\infty} M_k < \infty$, it is clear that there is a $k > N$ such that $|a_k(z)| < \frac{1}{2}$, and we have hypothesis (1). The theorem goes through as before simply with $M_k$ replaced by $2M_k$.

## 3.4 Examples

### 3.4.1 Convergent Example

Consider the product

$$F(z) = \prod_{k=1}^{\infty}(1 - \frac{z}{k})e^{\frac{z}{k}}$$

We then expand the function inside of the product, $(1 - \frac{z}{k})e^{\frac{z}{k}}$, and find that the $\frac{1}{k}$ term cancels

$$(1 - \frac{z}{k})e^{\frac{z}{k}} = (1 - \frac{z}{k})(1 + \frac{z}{k} + \frac{z^2}{2!k^2} + ...) = 1 - \frac{z^2}{2!k^2}$$

Additionally, the Taylor series of the *log* of the above function is as follows,

$$\begin{aligned}
\log((1-w)e^w) &= \log(1-w) + w \\
&= -(\frac{w^2}{2} + \frac{w^3}{3} + \frac{w^4}{4} + ...) \\
&= -(w^2)(\frac{1}{2} + \frac{w}{3} + \frac{w^2}{4} + ...)
\end{aligned}$$

hence for $|w| < \frac{1}{2}$

$$\begin{aligned}
|\log(1-w)e^w| &\leq |w|^2(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ...) \\
&= |w|^2(\frac{1}{2})(\frac{1}{1-\frac{1}{2}}) \\
&= |w|^2
\end{aligned}$$

Thus for $|z| < R$ and $k > 2R$, for any fixed value $R$

$$|\log(1 - \frac{z}{k})e^{\frac{z}{k}}| \leq |\frac{z}{k}|^2 \leq (\frac{R}{k})^2$$

so from the Weierstrass M-test, the original product converges uniformly to an entire function with simple zeros at $(1 - \frac{z}{k}) = 0$ for $k = 1, 2, ....$

# 4 Mittag-Leffler

## 4.1 What is a Mittag-Leffler expansion?

The Mittag-Leffler theorem, named after mathematician Gösta Mittag-Leffler, studies the existence of meromorphic functions with prescribed poles.[3] This expansion can be used to express meromorphic functions as a sum of partial fractions. As mentioned in the Significance and History section, the theorem is closely tied with the Weierstrass Factorization theorem, which demonstrates that there exists holomorphic functions with prescribed zeros that can be represented as an infinite product. The Mittag-Leffler theorem implies that for any meromorphic function f(z) in $\mathbb{C}$ with poles $a_n$ and corresponding principal part $g_n$(z) of the Laurent expansion of f(z) in a neighborhood of $a_n$ can be expanded in a series where the entire function h(z) is determined by f(z). [1]

## 4.2 Mittag-Leffler General Case

Mittag-Leffler did extensive research into meromorphic functions. A primary contribution to the study of infinite products was the Mittag-Leffler Expansion. This presents an alternate to Taylor and Laurent expansions.

### 4.2.1 Mittag-Leffler General Case

Let f(z) be a meromorphic function in the complex plane with poles $\{z_j\}$ and corresponding principal parts $\{p_j(z)\}$. Then there exist polynomials $\{g_j(z)\}_{j=1}^{\infty}$ such that:

$$f(z) = p_0(z) + \sum_{j=1}^{\infty}(p_j(z) - g_j(z)) + h(z) = \hat{f}(z) + h(z) \tag{3}$$

holds and the series $\sum_{j=1}^{\infty}(p_j(z) - g_j(z))$ converges uniformly on every bounded set not containing the points $\{z_j\}_{j=0}^{\infty}$. [5]

The part of equation (3) that is called h(z) is necessarily an entire function and $\hat{f}(z)$ has the same number of poles, strength and locations as $f(z)$.

### 4.2.2 Proof: Mittag-Leffler General Case Theorem

The proof for Mittag-Leffler General Case is extensive, however, the following is an outline of the proof as given by Mark J. Ablowitz and Athanassios S. Fokas.

Each of the principal parts $\{p_j(z)\}_{j=1}^{\infty}$ can be expanded in a convergent Taylor series (around z=0) for

$|z| < |z_j|$. It can be shown that enough terms can be taken in this Taylor series that the polynomials $g_j(z)$ obtained by truncation of the Taylor series of $p_j(z)$ at order $z^{K_j}$

$$g_j(z) = \sum_{k=0}^{K_j} B_{k,j} z^k \tag{4}$$

ensure that the difference $|p_j(z) - g_j(z)|$ is suitably small. It can be shown (e.g. Henrici, volume I, 1997) that for any $|z| < R$, the polynomials $g_j(z)$ of order $K_j$ ensure that the series

$$\sum_{j=1}^{\infty} |p_j(z) - g_j(z)| \tag{5}$$

converges uniformly. [5]

## 4.3 Mittag-Leffler Simple Poles Theorem

There is also a specific case of the Mittag-Leffler Expansion called Mittag-Leffler case for simple poles. The theorem is stated as such:

Let $a_j$ and $z_j$ be sequences where $z_j$ distinct, and $|z_j| \to \infty$ as $j \to \infty$ and $Q \in \mathbb{Z}$ such that.

$$\sum_{j=1}^{\infty} \frac{|a_j|}{|z_j|^{Q+1}} < \infty \tag{6}$$

## 4.4 Examples of Mittag-Leffler

Let's consider the function

$$f(z) = \pi \tan \pi z$$

This function has simple poles at $z_j = j - \frac{1}{2}, j = 0, \pm 1, \pm 2, ....$ The strength of these poles is $a_j = -1$ which can be attained from the Laurent Series of f(z) in the neighborhood of $z_j$. Let $z = \hat{z} + \frac{1}{2}$+j, then

$$\begin{aligned} f(z) &= \pi \frac{\sin(\frac{\pi}{2} + \pi j + \pi \hat{z})}{\cos(\frac{\pi}{2} + \pi j + \pi \hat{z})} \\ &= \pi \frac{\sin(\frac{\pi}{2} + \pi j)\cos(\pi \hat{z})}{-\sin(\frac{\pi}{2} + \pi j)\sin(\pi \hat{z})} \\ &= -\pi \frac{\cos(\pi \hat{z})}{\sin(\pi \hat{z})} = -\pi \cot(\pi \hat{z}) \\ &= \frac{-\pi(1 - \frac{(\pi \hat{z})^2}{2!} + ...)}{\pi(\hat{z} - \frac{(\pi \hat{z})^3}{3!} + ...)} \\ &= \frac{-1}{\hat{z}}(1 - \frac{1}{3}(\pi \hat{z})^2 + ...) = -\pi \cot(\pi \hat{z}) \end{aligned} \tag{7}$$

The function f($\hat{z}$) has simple pole at $\hat{z}_j = j$. The principal part at each $z_j$ is therefore given by $p_j(z) = \frac{-1}{z-\frac{1}{2}-j}$. Then the series from the Mittag-Leffler Simple Poles theorem:

$$\sum_{j=-\infty, j\neq 0}^{\infty} \frac{-1}{|j|^{Q+1}}$$

converges for $Q = 1$. Consequently from the Mittag-Leffler General Case theorem and the equation:

$$f(z) = p_0(z) + \sum_{j=1}^{\infty} (\frac{a_j}{z_j}) L(\frac{z}{z_j}, m) + h(z)$$

where L is:

$$L(w, m) = \frac{1}{w-1} + 1 + w + w^2 + ... + w^{m-1}$$

the general form of the function is fixed to be:

$$\begin{aligned}
\pi \tan \pi z &= \frac{-1}{z-\frac{1}{2}} - \sum_{j=-\infty}^{\infty}{}' (\frac{1}{(z-j-\frac{1}{2})} + \frac{1}{j}) + h(z) \\
&= -2 \sum_{j=0}^{\infty} \frac{z}{z^2 - (j-\frac{1}{2})^2} + h(z) \\
&= -\frac{1}{\hat{z}} - 2 \sum_{j=-\infty}^{\infty} \frac{\hat{z}}{\hat{z}^2 - j^2} + h(\hat{z}) \\
&= - \sum_{j=-\infty}^{\infty} \frac{\hat{z}}{\hat{z}^2 - j^2} + h(\hat{z}) = -\pi \cot(\pi \hat{z})
\end{aligned} \tag{8}$$

where the prime in the first sum means that the term $j = 0$ is excluded and where $h(z)$ is an entire function. Note that the $(1/j)$ term in equation (8) is a necessary condition for the series to converge.

Equation (8) holds with h(z)=0 because of:

$$I = \frac{-1}{2\pi i} \oint_C \pi \cot \pi \zeta (\frac{1}{z-\zeta} + \frac{1}{\zeta}) d\zeta$$

where C is an appropriate closed contour.

This example continues in the Weierstrass Factorization theorem example.

# 5 Weierstrass Factorization Theorem

## 5.1 What is Weierstrass Factorization Theorem?

The Weierstrass Factorization theorem, named after Karl Weierstrass, asserts that an entire function can be represented by a product involving its zeros. In addition, every sequence tending to infinity has an associated entire function with zeros at precisely the points of that sequence. This section will focus on the second form of the theorem, which extends to meromorphic functions and allows one to consider a given meromorphic function as a product of three factors: terms depending on the function's poles and zeros, and an associated non-zero holomorphic function.

## 5.2 Statement

Let $f(z)$ be a meromorphic function and $z_0 = 0$ be an isolated zero of $f(z)$ with multiplicity $m \geq 0$. Let $(z_j)_{j=1}^{\infty}$ be a sequence which consist of all zeros or poles of $f(z)$ with respective multiplicities $a_j$ such that $|z_j| > 0$ and $|z_j| \leq |z_{j+1}|$, where $|z_j| \to \infty$ as $j \to \infty$.
Then there exists a holomorphic function $g(z)$ and a sequence of integers $\{Q_j(z)\}_{j=1}^{\infty}$ such that.

$$f(z) = z^m \prod_{j=1}^{\infty} \Big[(1 - \frac{z}{z_j}) exp(\sum_{k=0}^{Q_j - 1} \frac{(\frac{z}{z_j})^{k+1}}{k+1})\Big]^{a_j} e^{g(z)} \tag{9}$$

We define $Q_j$ as any sequence of integers such that for all $|a_j| > 0$ the Mittag-Leffler Simple Poles theorem is satisfied.

$$\sum_{j=1}^{\infty} \frac{|a_j|}{|z_j|^{Q_j+1}} < \infty \tag{10}$$

## 5.3 Simplified Statement

It is important to note that $z_j$ cannot have a limit point other than infinity. By means of illustration, if $z_j$ has a limit point $z_1$, then $z_j$ can only be taken arbitrarily close to $z_1$. Consequently, $f(z)$ would not have isolated zeros, and would not be entire, resulting in a contradiction. When equation (10) diverges, it can be helpful to represent $f(z)$ with the following more general equation using Weierstrass elementary factors, $E_{Q_j}$.

$$f(z) = z^m \prod_{j=1}^{\infty} E_{Q_j}(\frac{z}{z_j}) e^{g(z)} \quad \text{where} \quad E_{Q_j} = \begin{cases} (1-z) & j = 0 \\ (1-z) exp(z + \frac{z^2}{2} + ... + \frac{z^j}{j}) & \text{else} \end{cases} \tag{11}$$

## 5.4 Proof

Let $f(z)$ be the same $f(z)$ from equations (9) and (11), and let $h(z)$ be a meromorphic function with the same zeros and poles as $f(z)$ counting multiplicity, such that $z_0 = 0$ is a zero or pole with multiplicity $m \geq 0$ and the sequence $(z_j)$ is all zeros or poles with multiplicity $a_j$ such that $|z_j| > 0$, $|z_j| \to \infty$ as $j \to \infty$. We define $h(z)$ as the following equation [3].

$$h(z) := exp(\int_0^z f(t)dt) \quad \text{where} \quad f(t) = \sum_{j=1}^{\infty} (\frac{t}{z_j})^{Q_j} \frac{a_j}{t - z_j}$$

The function $f(t)$ is a Mittag-Leffler expansion and is therefore uniformly convergent. Thus, on a path from 0 to z not containing $z_j = z_1, z_2, ...$ term-by-term integration is allowed.

$$\begin{aligned}
h(z) &= exp(\int_0^z \sum_{j=1}^{\infty} (\frac{t}{z_j})^{Q_j} \frac{a_j}{t - z_j} dt) \\
&= \prod_{j=1}^{\infty} exp(\int_0^z (\frac{a_j}{t - z_j} + \frac{a_j}{z_j} \frac{(t/z_j)^{Q_j} - 1}{(t/z_j) - 1}) dt \\
&= \prod_{j=1}^{\infty} exp(\int_0^z (\frac{a_j}{t - z_j} + \frac{a_j}{z_j} \sum_{k=1}^{Q_j} (\frac{t}{z_j})^{k-1}) dt \\
&= \prod_{j=1}^{\infty} exp\big(\log(1 - \frac{z}{z_j})^{a_j} + a_j \sum_{k=1}^{Q_j} (\frac{1}{k})(\frac{z}{z_j})^k\big) \\
&= \prod_{j=1}^{\infty} (1 - \frac{z}{z_j})^{a_j} exp\big(a_j \sum_{k=1}^{Q_j} \frac{1}{k} (\frac{z}{z_j})^k\big) \\
&= \prod_{j=1}^{\infty} \Big[(1 - \frac{z}{z_j}) exp(\sum_{k=0}^{Q_j - 1} \frac{(\frac{z}{z_j})^{k+1}}{k+1})\Big]^{a_j}
\end{aligned}$$

The entire function $h(z)$ can be simplified with the equation below, using Weierstrass elementary factors $E_{Q_j}$.

$$h(z) = \prod_{j=1}^{\infty} E_{Q_j}(\frac{z}{z_j}) \quad \text{where} \quad E_{Q_j} = \begin{cases} (1 - z) & j = 0 \\ (1 - z) exp(z + \frac{z^2}{2} + ... + \frac{z^j}{j}) & \text{else} \end{cases}$$

Because $f(z)$ and $h(z)$ are meromorphic and share the same zeros and multiplicities, it follows that $\frac{f(z)}{h(z)} \neq 0$ is holomorphic. Since $\frac{f(z)}{h(z)}$ is both holomorphic and nowhere zero there exists a holomorphic function $g(z)$

such that $g(z) = \log(\frac{f(z)}{h(z)})$. Rearranging this equation we get,

$$f(z) = e^{g(z)}h(z)$$

The entire function $f(z)$ has a zero of multiplicity $m \geq 0$ or a pole of multiplicity $-m$ at $z_0 = 0$, so $z^{-m}f(z)$ is regular and not equal to zero. Thus, $f(z)$ can be represented by the following equation, which is the Weierstrass Factorization.

$$f(z) = z^m \prod_{j=1}^{\infty} E_{Q_j}(\frac{z}{z_j})e^{g(z)} \quad \text{where} \quad E_{Q_j} = \begin{cases} (1-z) & j = 0 \\ (1-z)exp(z + \frac{z^2}{2} + ... + \frac{z^j}{j}) & \text{else} \end{cases}$$

## 5.5 Examples of Weierstrass

This example will demonstrate how to find the Weierstrass Factorization of $\cos(\pi z)$ from its corresponding Mittag-Leffler expansion. We begin by considering the Mittag-Leffler expansion for $\pi\tan(\pi z)$ from section 4 because it is easiest to evaluate the Mittag-Leffler expansion for $\frac{f'(z)}{f(z)}$ where $f(z) = \cos(\pi z)$.

$$\pi \tan \pi z = -2\sum_{j=0}^{\infty} \frac{z}{z^2 - (j-\frac{1}{2})^2} + h(z) \tag{12}$$

Let $h(z) = 0$ in equation (12) because $h(z) = 0$ is an entire function. The equation can then be integrated using the principal branch of the log.

$$\begin{aligned}
\int \pi\tan(\pi z)dz &= \int -2\sum_{j=0}^{\infty} \frac{z}{z^2-(j-\frac{1}{2})^2}dz \\
-\log(\cos(\pi z)) &= \sum_{j=0}^{\infty}\int \frac{-2z}{z^2-(j-\frac{1}{2})^2}dz \\
-\log(\cos(\pi z)) &= -\sum_{j=0}^{\infty}\left(\log(z^2-(j-\frac{1}{2})^2) - C_n\right) \quad \text{where } C_n \text{ is a constant} \\
\log(\cos(\pi z)) &= \sum_{j=0}^{\infty}\left(\log(z^2-(j-\frac{1}{2})^2) - C_n\right) \\
\cos(\pi z) &= exp\sum_{j=0}^{\infty}\left(\log(z^2-(j-\frac{1}{2})^2) - C_n\right)
\end{aligned}$$

Now, one must solve for $C_n$, where $C_n$ is a convergence factor, so let $C_n = \log((j - \frac{1}{2})^2))$.

$$\begin{aligned}
\cos(\pi z) &= exp\sum_{j=0}^{\infty}\big(\log(z^2 - (j - \frac{1}{2})^2) - \log((j - \frac{1}{2})^2))\big) \\
&= exp\sum_{j=0}^{\infty}\log(\frac{(z^2 - (j - \frac{1}{2})^2)}{(j - \frac{1}{2})^2}) \\
&= exp\sum_{j=0}^{\infty}\log(1 - (\frac{z}{j - \frac{1}{2}})^2) \\
&= \prod_{j=0}^{\infty}(1 - (\frac{z}{j - \frac{1}{2}})^2)
\end{aligned}$$

Thus, $\cos(\pi z) = \prod_{j=0}^{\infty}(1 - (\frac{z}{j-\frac{1}{2}})^2)$, which is a Weierstrass Factorization.

## 6 Applications

Though the aforementioned theorems have significant applications in complex analysis, they also are widely applicable to certain areas of physical and applied sciences. Almost a century after Mittag-Leffler theorem was discovered, this function has come into prominence in physical, biological, engineering, and earth sciences. These theorems have applications in fluid flow, rheology, diffusive transport similar to diffusion, electric networks, probability, and statistical distribution theory. For applied sciences, Mittag-Leffler theorem is used for deriving kinetic equations, and studying time-fractional diffusion and fractional-space diffusion. In addition to the inverse LaPlace transform, it is also useful for studying non-linear waves. [8] The Mittag-Leffler theorem is also used for sampling discontinuous signals for digital signal processing in order to eliminate the error of transformations between continuous-time and discrete-time systems. [6]

Furthermore, the Weierstrass Factorization theorem could be considered an application of the Mittag-Leffler theorem. Even though the Weierstrass Factorization theorem was published first by Karl Weierstrass, it is possible to prove the Weierstrass Factorization theorem by means of the Mittag-Leffler as demonstrated in section 5 of this paper. Consequently, the two theorems are closely related and allow one to represent a meromorphic function as either a sum of partial fractions or an infinite product.

## 7 Conclusion

This paper proved and discussed the significance of three closely related theorems in the field of complex analysis: Weierstrass M-test, Mittag-Leffler, and the Weierstrass Factorization. To begin with, the Weierstrass M-test allows one to determine if a product of functions is uniformly convergent, which allows one to prove Mittag-Leffler. The Mittag-Leffler theorem concerns the existence of meromorphic functions with prescribed poles. It can be used to express any meromorphic function as a sum of partial fractions. Similarly, the Weierstrass Factorization theorem states that a meromorphic function can be expressed by a product involving its zeros. The Mittag-Leffler theorem and the Weierstrass Factorization are sometimes referred to as sister theorems because these theorems in tandem allow one to express any meromorphic function as either a sum of partial fractions or an infinite product involving its zeros, which can be very useful.

If given more time, our group would like to explore the many applications of these theorems in more detail. In particular, the derivation of kinetic equations is intriguing. We would also like to look into Hadamard's Factorization theorem [4], which is a more precise version of the Weierstrass Factorization. One can utilize Hadamard's Factorization when some information is known about the growth of the function. Extensive knowledge of this theorem would allow us to give more precise infinite product expressions of meromorphic functions.

## References

[1] https://www.encyclopediaofmath.org/index.php/Mittag-Leffler_theorem, 2018.

[2] College algebra. http://web.maths.unsw.edu.au/~iand/5685/week5.pdf, 2018.

[3] Historical overview of the mittag-leffler functions. http://www.springer.com/cda/content/document/cda_downloaddocument/9783662439296-c2.pdf?SGWID=0-0-45-1482936-p176811612, 2018.

[4] Lecture notes in mathematics. http://www.math.tamu.edu/~boas/courses/618-2016a/notes2016.pdf, 2018.

[5] Mark J Ablowitz and Athanassios S Fokas. *Complex variables*. Cambridge Univ. Press, 2011.

[6] Michael J. Corinthios. Application of the mittag-leffler expansion to sampling discontinuous signals. *IET Signal Processing*, 7(9):863–878, 2013.

[7] Laura E. Turner. The mittag-leffler theorem: The origin, evolution, and reception of a mathematical result, 1876-1884, 2018.

[8] H. J. Haubold, A. M. Mathai, and R. K. Saxena. Mittag-leffler functions and their applications. *Journal of Applied Mathematics*, 2011:1–51, 2011.